THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2058 Honours Mathematical Analysis I
Suggested Solutions for HW3

1. Suppose (x_n) is a bounded sequence of real numbers. Define

   (a) $L_1 := \sup_{n\in\mathbb{N}} \inf_{k\geq n} x_k$;

   (b) $L_2 := \sup\{w \in \mathbb{R} : x_m < w \text{ for at most finitely many } m\}$;

   (c) $L_3 := \inf S$ where S denotes the set of sub-sequential limit of (x_n).

   Show that $L_1 = L_2 = L_3$.

   Solution. We first show that $L_1 = L_2$. Let

   $$W = \{w \in \mathbb{R} : x_m < w \text{ for at most finitely many } w\}.$$

   Then we want to show that $L_1 = \sup W$. We first show that L_1 is an upper bound of W. Let $w \in W$. Then by definition $x_m < w$ for at most finitely many m. Equivalently, this means there is an $N_w \in \mathbb{N}$ such that for all $k \geq N_w$, $w \leq x_k$. So we have

   $$w \leq \inf_{k\geq N_w} x_k \leq x_k.$$

   Then taking supremum over N_w, we also have that $w \leq L_1$. Now let $\varepsilon > 0$. We now want to show that $L_1 - \varepsilon \in W$. By definition of L_1, there is an $N_\varepsilon \in \mathbb{N}$ such that for any $k \geq N_\varepsilon$, we have

   $$L_1 - \varepsilon < x_k.$$

   Then there are only at most finitely many m such that $x_m < L_1 - \varepsilon$ and so $L_1 - \varepsilon \in W$. So $L_1 = L_2$.

   We now show that $L_2 = L_3$. We first show that L_3 is an upper bound of W. Suppose not, then there is an $w \in W$ such that $L_3 < w$, i.e. there is an $\ell < w$ such that there is a subsequence (x_{n_k}) of (x_n) with $\lim_{k\to+\infty} x_{n_k} = \ell < w$. But then this gives infinitely many k such that $x_{n_k} < w$, contradicting the fact that $w \in W$. So L_3 must be an upper bound of W. Let $\varepsilon > 0$. Then we finally want to show that $L_3 - \varepsilon \in W$, that is, we want to show that $x_m < L_3 - \varepsilon$ for at most finitely many m. Suppose not, then there is a subsequence (x_{n_m}) of (x_n) with $x_{n_m} < L_3 - \varepsilon$. Since this subsequence is bounded, by the Bolzano-Weierstrass Theorem, it contains a convergent subsequence $x_{n_{m_k}} \to \ell$, say. Since taking limits preserves order, we have $\ell < L_3 - \varepsilon$, but we have found a sub-sequential limit of x_n which is strictly less than L_3 a contradiction. Hence $L_3 - \varepsilon \in W$ and we conclude that $L_2 = L_3$ as well. ◀

2. Suppose (x_n) is a sequence of positive real number. Show that

   $$\limsup_{n\to+\infty} x_n^{1/n} \leq \limsup_{n\to+\infty} \frac{x_{n+1}}{x_n}$$

Solution. Let $u := \limsup_{n\to+\infty} \frac{x_{n+1}}{x_n}$. Note that since $x_n > 0$ for each n, $u > 0$. Using the definition of $\limsup_{n\to+\infty} a_n := \inf_n \sup_{k\geq n} a_k$ for the sequence $\frac{x_{n+1}}{x_n}$, we have that for all $\varepsilon > 0$, there is an $n \in \mathbb{N}$ such that

$$\frac{x_{k+1}}{x_k} \leq u + \varepsilon.$$

Then we have that

$$\frac{x_k}{x_n} = \frac{x_{n+1}}{x_n}\frac{x_{n+2}}{x_{n+1}}\cdots\frac{x_k}{x_{k-1}} = \prod_{j=n}^{k-1}\frac{x_{j+1}}{x_j} \leq (u+\varepsilon)^{k-1-n}.$$

So we have that

$$x_k \leq (u+\varepsilon)^{k-1-n}x_n \Leftrightarrow x_k^{1/k} \leq \left(\frac{x_n}{(u+\varepsilon)^{1+n}}\right)(u+\varepsilon).$$

Then since $a^{1/k} \to 1$ as $k \to +\infty$, taking $k \to +\infty$, we have that

$$\limsup_{k\to+\infty} x_k^{1/k} \leq u + \varepsilon.$$

Since ε was taken arbitrarily, we have the desired result.

To show an example of strict inequality, consider the sequence

$$x_n = \begin{cases} 1, & n \text{ even} \\ 2, & n \text{ odd} \end{cases}.$$

Then $\frac{x_{n+1}}{x_n}$ is either $\frac{1}{2}$ or 2, and so $\limsup_{n\to+\infty}\frac{x_{n+1}}{x_n} = 2$, but $\limsup_{n\to+\infty} x_n^{1/n} = 1$. ◀

3. Show that if (x_n) is a unbounded sequence, then there exists a subsequence (x_{n_k}) such that $x_{n_k}^{-1} \to 0$ as $k \to +\infty$.

Solution. Since $\{x_n\}$ is unbounded, for any $M > 0$, we can find an $n \in \mathbb{N}$ such that $|x_n| > M$. We construct the desired subsequence by induction. By unboundedness, we pick $n_1 \in \mathbb{N}$ such that $|x_{n_1}| > 1$. We then pick $n_2 \in \mathbb{N}$ such that

$$|x_{n_2}| > \max\{2, |x_1|, |x_2|, \ldots, |x_{n_1}|\},$$

so that $\left|\frac{1}{x_{n_2}}\right| < \frac{1}{2}$ and $n_2 > n_1$.

Now suppose $n_1 < n_2 < \cdots < n_k$ are chosen so that $\left|\frac{1}{x_{n_\ell}}\right| < \frac{1}{\ell}$ for $1 \leq \ell \leq k$. Then we pick $n_{k+1} \in \mathbb{N}$ so that

$$|x_{n_{k+1}}| > \max\{k+1, |x_1|, |x_2|, \ldots, |x_{n_k}|\}$$

then by the inductive hypothesis, we have that $\left|\frac{1}{x_{n_{k+1}}}\right| < \frac{1}{k+1}$ and that $n_{k+1} > n_k$. Then the subsequence we have produced has limit 0 by the Squeeze theorem. ◀

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4. Suppose every subsequence of (x_n) has a subsequence converging to 0, show that $x_n \to 0$.

Solution. Suppose for the sake of contradiction that x_n does not converge to 0. Then there is an $\varepsilon_0 > 0$ such that for all $n \in \mathbb{N}$ there is an $n_k \geq n$ with

$$|x_{n_k}| \geq \varepsilon_0.$$

Note that n_k may not be increasing with respect to k (and hence (x_{n_k}) is not actually a subsequence. However, we can pick out an increasing sequence by inductively defining $m_1 = n_1$ and $m_k = n_{m_{k-1}}$. Then by construction $m_k > m_{k-1}$ and the subsequence (x_{m_k}) does not converge to 0. But this contradicts the fact that x_{m_k} contains a subsequence that does converge to 0, and hence we are done. ◀

5. If $x_1 < x_2$ and $x_n = \frac{1}{4}x_{n-1} + \frac{3}{4}x_{n-2}$ for $n > 2$. Show that (x_n) is convergent. Find its limit.

Solution. Note that

$$|x_{n+2} - x_{n+1}| = \left|\frac{1}{4}x_{n+1} + \frac{3}{4}x_n - x_{n+1}\right| = \left|\left(-\frac{3}{4}\right)(x_{n+1} - x_n)\right|.$$

So (x_n) is a contractive sequence and hence is Cauchy and is convergent. We have

$$x_{n+2} - x_{n+1} = -\frac{3}{4}(x_{n+1} - x_n) = \left(-\frac{3}{4}\right)^2 (x_n - x_{n-1}) = \cdots = \left(-\frac{3}{4}\right)^n (x_2 - x_1)$$

Summing up the expression, we have

$$\sum_{k=0}^{n}(x_{k+2} - x_{k+1}) = (x_2 - x_1)\sum_{k=0}^{n} \frac{1 - \left(-\frac{3}{4}\right)^{n+1}}{1 - \left(-\frac{3}{4}\right)}$$

$$x_{n+2} - x_1 = \frac{4}{7}(x_2 - x_1)\left(1 - \left(-\frac{3}{4}\right)^{n+1}\right).$$

Taking $n \to +\infty$ and since $\lim_{n\to+\infty} \left(-\frac{3}{4}\right)^{n+1} = 0$, we have

$$\lim_{n\to+\infty} x_{n+2} = x_1 + \frac{4}{7}x_2 - \frac{4}{7}x_1 = \frac{3}{7}x_1 + \frac{4}{7}x_2.$$

◀

6. Let $p \in \mathbb{N}$, give an example of sequence (x_n) that is not Cauchy but satisfies $|x_{n+p} - x_n| \to 0$ as $n \to +\infty$.

Solution. Let $x_n := \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$ which is not Cauchy because it is divergent. But we have that for any $p \in \mathbb{N}$, we have

$$0 < x_{n+p} - x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p} \leq \frac{p}{n+1}$$

and $\lim_{n\to+\infty} \frac{p}{n+1} = 0$, so by the Squeeze Theorem, $|x_{n+p} - x_n| \to 0$ as $n \to +\infty$. ◀